Feb. 3, 2014

Today: A first pass at the antiderivative (which we will later call an integral)

The Antiderivative

$$f(x) \underset{\text{antiderivative}}{\overset{\text{derivative}}{\rightleftarrows}} f'(x)$$

If $\frac{d}{dx} f(x) = 2x$, what is $f(x)$?

$f(x) = x^2$ works

also, $f(x) = x^2+1$ works...

If $\frac{d}{dx} g(x) = 3x^2+1$, what is $g(x)$?

$f(x) = x^3+x$ works

so does x^3+x+1

$x^3+x+\pi$

$\underbrace{x^3+x+1018 \ldots}_{\text{we'll get back to this in a minute}}$

<u>DEF</u>: An <u>antiderivative</u> of a function f on an interval I is another function F s.t. $F'(x) = f(x)$ for all x in I.

<u>Example</u>: $f(x) = 2x$ an antiderivative is $F(x) = x^2$

another is $F(x) = x^2+20$

There are <u>lots</u> of antiderivatives which look like

$$F(x) = x^2 + \underbrace{C}_{\text{constant}}$$

We can describe all antiderivatives w/ $+C$.

For instance, if $F(x)$ is an antider of $f(x)$ then $F(x)+\underbrace{C}_{\text{any constant.}}$ describes all antiders of $f(x)$.

Why? Say $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$.

$$(F'(x)=f(x),\ G'(x)=f(x))$$

$$\frac{d}{dx}(F(x)-G(x)) = F'(x)-G'(x) = f(x)-f(x) = 0$$

so $F(x)-G(x)$ is a constant.

All antiderivatives differ by a constant.

The General Antiderivative

We call $F(x)+C$ the general antiderivative or (later) indefinite integral.

We write:

$$F(x)+C = \int f(x)\,\underbrace{dx}$$

this notation will be more motivated later on. For now dx tells us that x is the variable.

Ex: $\int 2x\,dx = x^2+C$

Ex: $\int x^2 dx = \frac{1}{3}x^3+C$ b/c $\frac{d}{dx}\left(\frac{1}{3}x^3+C\right) = 3\cdot\frac{1}{3}x^2 = x^2$

Ex: $\int x^3 dx = \frac{1}{4}x^4+C$ b/c $\frac{d}{dx}\left(\frac{1}{4}x^4+C\right) = 4\cdot\frac{1}{4}x^3+C = x^3$

Ex: $\int x^{-3}\,dx = -\frac{1}{2}x^{-2} + C$ b/c $\frac{d}{dx}\left((-\frac{1}{2})x^{-2} + C\right) =$

$-2 \cdot -\frac{1}{2}x^{-3} = x^{-3}$

Pattern?

$$\boxed{\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C}$$

Some Antiderivatives:

- $\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C$
- $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x$
- $\int e^x\,dx = e^x + C$
- $\int \sec^2 x\,dx = \tan x + C$

B/c $\int$ is opposite of $\frac{d}{dx}$ we get some propertives

(1) If a is a constant, $f(x)$ has an antider. then

$$\int a f(x)\,dx = a\left(\int f(x)\,dx\right)$$

(2) f, g both have antiderivatives

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

In general, for constants a, b

$$\int (a f(x) + b g(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

Differential Equations:

A diff. equ. is an equation involving derivatives.

Goal: Solve for y (usually)

A Sol'n: A function you can plug in that satisfies equation.

Ex: $\frac{dy}{dx} - 5y = 0$

try $y = e^{5x}$. $\frac{dy}{dx} = \frac{d}{dx} e^{5x} = 5e^{5x}$

$\frac{dy}{dx} - 5y = 5e^{5x} - 5(e^{5x}) = 0$ ✓

Antiderivatives are the simplest diff. equations:

Ex: (1) $\frac{d}{dx}y = x^2+1$

$$y = \int (x^2+1)\,dx$$

$$\boxed{y = \frac{1}{3}x^3 + x + C}$$ General Solution

Observe $\frac{d}{dx}\left(\frac{1}{3}x^3 + x + C\right) = x^2+1$ ✓

Initial Value Problems

- when you solve a diffy Q (differential equation) you get a general sol'n (with C).
- In an initial value Problem (IVP) you get additional information, so there won't be a C in your answer.

Ex: (IVP) Find a solution to the diffy Q $\frac{d}{dx}y = x^2+1$ which also satisfies $\underbrace{y(2) = 8/3}_{\text{when } x=2,\ y=8/3}$

General Sol'n: $y = \frac{1}{3}x^3 + x + C$ (show some general sol'ns on graph)

* In an IVP, you solve for C as well!*

use $y(2) = 8/3$: $8/3 = \frac{1}{3}(2)^3 + 2 + C$

$$8/3 = 8/3 + 2 + C$$

$$-2 = C$$

$$\boxed{\text{Particular Sol'n: } y = \frac{1}{3}x^3 + x - 2}$$

Ex: Solve IVP: $\begin{cases} \frac{dy}{dx} = 2x + \sin x \\ y(0) = 0 \end{cases}$ (sometimes write y' instead of dy/dx)

(1) general sol'n: $\int (2x + \sin x)dx = \int 2x\,dx + \int \sin x\,dx$

$$y = x^2 - \cos x + C$$

(2) Solve for C: $0 = 0^2 - \cos(0) + C$

$$0 = -1 + C$$
$$1 = C$$

(3) Particular Sol'n: $y = x^2 - \cos x + 1$

Ex: Solve initial value problem:

$$\begin{cases} y'' = \cos x \\ y'(\pi/2) = 2 \\ y(\pi/2) = 3\pi \end{cases}$$

Goal: Solve for y - will need to take antiderivative twice.

- $y'' = \cos x$

  $y' = \int \cos x\,dx = \sin x + C$

*Now $y' = \sin x + C$. Since $y'(\pi/2) = 2$, we can solve for C.

- $y' = \sin x + C$, so $2 = \sin(\pi/2) + C$

  $1 = C$

Result: $y' = \sin x + 1$

- Solve for y: $y = \int (\sin x + 1)dx$

  $y = -\cos x + x + D$

- $y(\pi/2) = 3\pi$ so $3\pi = -\cos(\pi/2) + (\pi/2) + D$

  $3\pi - \pi/2 = D$

  $5\pi/2 = D$

- Particular sol'n: $\boxed{y = -\cos x + x + \frac{5\pi}{2}}$

Word Problems:

Worksheet